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THE PRESENTATION OF THE NOTION OF FUNCTION.

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In a previous article[1] I have shown how the work of the freshman year in colleges can be unified by presenting trigonometry, college algebra, and analytics from the point of view of the theory of functions. I argued further that, because the notion of functional independence is always considered somewhere in every one of the three courses, it would be advisable to treat it once for all at the very beginning of the year. Admitting this, I wish to show in the present article how the notion of function can be developed briefly, but so thoroughly that the student will think of it as something else besides axes, coördinates, and curves, and so comprehensively that the work will be useful regardless of whether the course is trigonometry, algebra, or analytics.

An examination of the textbooks shows that the usual procedure is about the following: (*a*) We have the explanation and illustration of the dependence of one variable upon another; (*b*) Denoting a value of one variable by x and of the other by y, we mark values of x and y on two perpendicular lines; (*c*) Assigning a value to x and computing the value of y, we determine a point in the plane, and the totality of these points defines a curve. Thereafter the student associates a curve with every equation in x, y, but he is most likely to think that the equation is a baptismal name of the curve. Another objection to this presentation is that coördinates are introduced too early, and that consequently the emphasis falls on "the point on the curve" instead of on "the two points on the axes." And further, after coördinates have been introduced, the essential idea of the relation between the two variables, represented now by abscissas and ordinates, is neglected.

The method I advocate aims to bring this idea of relationship between variables into the foreground. I would first present mathematics as a language

[1] The Unification of Freshman Mathematics, in this MONTHLY, April, 1916, page 101.

which facilitates the stating of these relations; second, I would emphasize the correspondence; and lastly, only after the notion of function and correspondence is grasped, would I explain the language of coördinates. Psychologically this can be the only correct order inasmuch as we present problems first and then study the common elements, coördinates, underlying the different problems. Under I, II, and III below, I sketch the process in so far as it differs from the conventional.

I. With the customary illustrations we show that the object of every branch of science is to find relations between variables. The doctor wishes to find the relation between the severeness of a typhoid epidemic and the number of bacteria in drinking water; the psychologist seeks to explain the relations between our variable experiences and emotions; the physicist learns experimentally how the length of an iron bar depends upon its temperature; the botanist relates plant forms to their climate; the astronomer finds that the size or mass of a planet influences the positions of other planets; the economists find that prosperity may depend upon business credit.

In order to state the relations in as concise a way as possible mathematicians have devised symbols (such as $+$, $-$, $\times$, $\div$, $=$, $\int$, ∂, $\surd$, ∇, $\sim$, ∞) which are so unique as to form almost a distinct language. The student must learn to use these symbols and be able to translate any statement in English words into these symbols much as he learns to translate English into French. Thus, the statement "one variable is always equal to twice another" is written mathematically "$y = 2x$," wherein y is a symbol for one variable and x is a symbol for the other. *The equation is a mathematical statement of an idea or law.* Its usefulness is due in part to its brevity and conciseness.

After stating the relation as simply as possible, mathematics aims to picture the relation by lines and curves. These pictures we call scales and graphs.

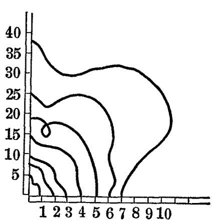

FIG. 1.

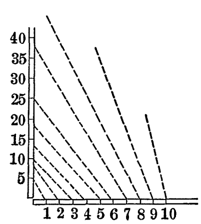

FIG. 2.

II. Assuming that the nature of scales has been explained, we can next place two scales parallel to each other, and then note that opposite numbers on the two scales correspond. Further, on these scales the *numbers* alone are the important things in the figure. In fact we use the figure only to show what number on one scale corresponds to another number on the other scale. Hence, we can do either of two things:

(*a*) Omitting the lines, and writing the numbers vertically, we can show the relation by what we call *data* or a table of corresponding values.

(*b*) We can place the two scales at right angles and show the correspondence, as in Figs. 1 or 2.

The teacher is then ready to explain Figs. 3 and 4. The student should be led to see the advantages of Fig. 4: from it we can approximate new corresponding values that would be impossible from a table of data or from Fig. 2 or 3. A figure like 1 may look ludicrous on the blackboard, and certainly no textbook has yet dared to contain one, but no one will deny that it illustrates a significant step in the evolution of the graph. The teacher must make sure that the student understands that the two lines (axes, as we later call them)

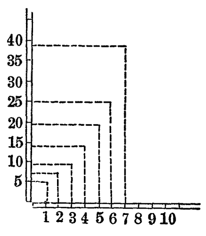

FIG. 3.

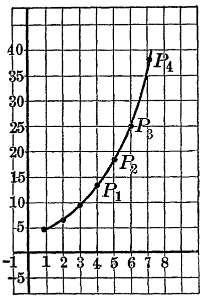

FIG. 4.

need not intersect at the zero point; that with a change of units the graph will look different, that if the problem hinges on the correspondence only over a particular range then the zeros may not even appear on either of the axes. All such matters as tend to emphasize *correspondence* should be considered at this point, and considered fully. It may even be useful to illustrate discontinuous correspondences, multiple valued ones, etc. Further, either here or before beginning II, the student should have some drill in finding functional relations, *i. e.*, stating by an equation a relation given descriptively. Problems involving motion or the measurement of physical magnitudes are the simplest and most useful.

If presenting the notion of function were all that we wish to do we could satisfactorily stop here. And we note that we are able to explain this concept without referring specifically to axes or coördinates. In truth, I believe the value of this mode of presentation lies in the very fact that coördinates and graphing have been kept in the background while the student is studying relations between variables. This is the reverse of what the textbooks do, for they consider coördinates first and then later, preliminary to locus problems, study functional relations. This reversal of order has probably arisen through the teaching of graphing in the high schools. Graphing is a very easy idea for the high school student to grasp; it enlightens the subject of simultaneous linear equations, and the teacher is glad to introduce a geometric idea into algebraic work. As a correlation of geometry and algebra such work is useful, but as an introduction to the notion of function it is misleading.

We can, however, now introduce coördinates in such a fashion that they will appear not at all as a *new* idea, but will appear as *that element which is common to all* the previous exercises. Pedagogically the introduction of coördinates at this stage is proper in that we now focus the student's attention on the underlying processes of his work. And since the student has been unconsciously using them, nothing further is needed except to define the technical terms by which the ideas will be referred to in the future.

III. We have used letters to denote one value from a set. We have axes, units, and points in the plane whose projections are the values which correspond. Conversely, every pair of corresponding values fixes a point in the plane. The numbers which determine the point we now choose to call coördinates. And thus we proceed to develop the rest of the terminology used in the study of relations $f(x, y) = 0$.

The work thus far can be done completely in three days, or even in two, depending upon how much the teacher wishes to drill on the various problems. All that is essential in explaining the notions of function, coördinates, graphing has been covered. If, however, an additional day is available, then some exercises involving the construction of such curves as $xy = 1$, $y = x^3$ are very profitable. While many papers have been published on the use of cross section paper for curve tracing I do not know that any one has pointed out its pedagogic value.

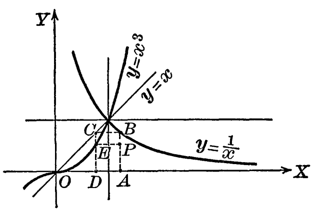

FIG. 5.

As a single illustration let us trace the thoughts of the student when asked for a method for constructing $x^3y = 1$ (see Fig. 5). He writes $y = 1/x^3$ and sees that he is dealing with the reciprocal of a variable and with the cube of a variable. Consequently he must first draw $y = 1/x$ and $y = x^3$. He selects any point A, writes $OA = x$, $AB = 1/x = DC = OD$, and finally $DE = \overline{OD}^3 = 1/x^3$. Consequently P is a point on the desired curve. The reader will see that this problem contains every idea associated with the notion of a function: the length of AB varies with the choice of A but varies according to a definite law; to cube AB it must be laid off horizontally from the origin; P and not E is the desired point on the curve because the ordinate is measured vertically from the end A of the abscissa; and finally, each equation of the problem states like a law the length of one line as compared with another. As the attention shifts from A to B to C, D, E, P the student must trace the fortunes of the variable as it is influenced by the different laws.

Whether the student thinks exactly in this fashion or not, this type of exercise is more valuable than the kind wherein he mechanically computes data for points and draws the curve through them. Even when the student can not himself discover the construction, nevertheless the subsequent reproduction of the teacher's method will be valuable if the student is required to give the reasons for each step. Also, this type of exercise more than any other affords good opportunity for questions by the teacher.